Linear Waves, Instabilities and Energy Principle

→ Contents

- this unit presents the linear structure, response theory and energetics for MHD

- proceed by: a.) linear waves
  b.) Least Action and Energy Principle
  c.) simple linear instabilities

- Later discuss nonlinear evolution, i.e.:
  i.e. a.) MHD shocks
  b.) collisionless shocks
  c.) MHD turbulence (later).

A.) Linear Waves in MHD → signal propagation

i.) Simple Cases

- before proceeding with full crank useful to discuss some limiting cases in depth
- always have $\underline{B_0} = B_0 \hat{z}$, $\rho = \rho_0$, $P = P_0$ } uniform

- consider

	$\nabla \cdot \underline{V} = 0$	$\nabla \cdot \underline{V} \neq 0$	
$\underline{k} = k\hat{z}$	Shear Alfvén	Acoustic	– parallel propagation
$\underline{k} = k\hat{x}$	X	Magnetosonic	– perpendicular propagation

→ $\underline{k} = k\hat{z}$, $\quad \nabla\cdot\underline{V} = 0$ Shear Alfven Wave

$$\left.\begin{aligned} \rho_0 \frac{\partial \tilde{\underline{V}}}{\partial t} &= -\nabla\left(\tilde{p} + \frac{\tilde{B}^2}{8\pi}\right) + \frac{\underline{B}_0\cdot\nabla\,\tilde{\underline{B}}}{4\pi} \\ \frac{\partial \tilde{\underline{B}}}{\partial t} &= \underline{B}_0\cdot\nabla\,\tilde{\underline{V}} \end{aligned}\right\} \text{linearized eqns.}$$

Now, $\nabla\cdot\tilde{\underline{V}} = 0 \Rightarrow$

$$-\nabla^2\left(\tilde{p} + \frac{\underline{B}_0\cdot\tilde{\underline{B}}}{8\pi}\right) + \underline{B}_0\cdot\nabla\cancelto{0}{(\nabla\cdot\tilde{\underline{B}})} = 0$$

$\left\{\rho_0, B_0\right.$ uniform

$$\therefore \quad \tilde{p} + \frac{\underline{B}_0\cdot\tilde{\underline{B}}}{8\pi} = 0$$

→ "perturbed pressure balance"

→ holds for incompressible (and weakly compressible) modes

$$\Rightarrow \quad \rho_0 \frac{\partial \tilde{\underline{V}}}{\partial t} = \frac{B_0}{4\pi}\frac{\partial}{\partial z}\tilde{\underline{B}}$$

$$\frac{\partial \tilde{\underline{B}}}{\partial t} = B_0 \frac{\partial}{\partial z}\tilde{\underline{V}}$$

$$\therefore \quad \boxed{\frac{\partial^2 \tilde{\underline{V}}}{\partial t^2} = \frac{B_0^2}{4\pi\rho_0}\frac{\partial^2}{\partial z^2}\tilde{\underline{V}}}$$

$B_0^2/4\pi\rho_0 = V_A^2$ Alfven velocity

$\Rightarrow \begin{cases} \omega^2 = k_\parallel^2 V_A^2 & \rightarrow \text{dispersion relation for } \underline{\text{shear}}\ \underline{\text{Alfven}}\ \underline{\text{wave}} \\ V_{ph} = V_{gr} = V_A \hat{z} & \rightarrow \text{speed} \begin{cases} \text{phase} \\ \text{group} \end{cases} \end{cases}$

wave propagates along $\hat{z}$ at Alfven speed

$\rightarrow$ wave is consequence of magnetic tension

$\frac{T}{M} \rightarrow \frac{B/4\pi}{\rho_0/B} \sim \frac{\text{tension-in-line}}{\text{mass-per-line}} \Rightarrow V_A^2$

$\rightarrow$ tension $\leftrightarrow$ plucking $\Rightarrow \tilde{\underline{V}} \perp B_0$

$\left(\begin{array}{c} \underline{\nabla} \cdot \underline{V} = 0 \\ \text{parallel variation} \end{array} \right)$

i.e. $\begin{cases} \tilde{\underline{V}} = \tilde{V}_x \hat{x} \\ \frac{\partial \tilde{\underline{B}}}{\partial t} = \frac{\partial}{\partial z} (\tilde{V}_x B_0) = \tilde{B}_x \hat{x} \end{cases}$

$\nabla \times (\underline{V} \times \underline{B}_0)$

$\partial_x \quad \partial_y \quad \partial_z$

$V_y B_0 \quad -V_x B_0$

$V_x \quad V_y \quad V_z$

in shear Alfven wave:

$\begin{cases} \tilde{\underline{V}}, \tilde{\underline{B}} \perp \underline{B}_0 \\ \tilde{\underline{V}} \parallel \tilde{\underline{B}}, \text{ but out of phase} \end{cases}$

→ energetics → construct "Poynting theorem"

$$\rho_0 \frac{\partial \tilde{\underline{V}}}{\partial t} = \frac{B_0}{4\pi} \frac{\partial}{\partial z} \tilde{\underline{B}} \qquad (1)$$

$$\frac{\partial \tilde{\underline{B}}}{\partial t} = B_0 \frac{\partial}{\partial z} \tilde{\underline{V}} \qquad (2)$$

∴ construct energy evolution

$$\mathcal{E} = \frac{\rho \tilde{V}^2}{2} + \frac{\tilde{B}^2}{8\pi}$$ → energy density

∴ (1)·$\underline{V}$ and (2)·$\underline{B}$ ⇒

$$\frac{\partial}{\partial t}\left(\rho_0 \frac{\tilde{\mathbf{V}}^2}{2} + \frac{\tilde{B}^2}{8\pi}\right) = \frac{B_0}{4\pi}\left(\underline{V} \cdot \frac{\partial \tilde{\underline{B}}}{\partial z} + \tilde{\underline{B}} \cdot \frac{\partial \underline{V}}{\partial z}\right)$$

∴

$$\frac{\partial}{\partial t}\left(\rho_0 \frac{\tilde{V}^3}{2} + \frac{\tilde{B}^2}{8\pi}\right) = \frac{B_0}{4\pi} \frac{\partial}{\partial z} (\tilde{\underline{V}} \cdot \tilde{\underline{B}})$$

and have Poynting form: $\frac{\partial \mathcal{E}}{\partial t} + \underline{\nabla} \cdot \underline{S} = 0$

$$\underline{S} = -\frac{B_0}{4\pi} (\underline{V} \cdot \underline{B})$$ → wave energy density flux

$\int d^3 \underline{v} \cdot \underline{B}$ → cross helicity

N.B. $\underline{S} = \frac{c}{4\pi} \underline{E} \times \underline{B}$, $\underline{P} = \underline{S}/c^2$

wave energy density flux (δ) ↳ wave momentum density

$\underline{E} = -\frac{\underline{V} \times \underline{B}_0}{c}$

$$\underline{S} = -\frac{1}{4\pi} (\tilde{\underline{V}} \times \underline{B}_0) \times \tilde{\underline{B}} = \frac{1}{4\pi}\left[\cancel{(\tilde{\underline{B}} \cdot \underline{B}_0)} \tilde{\underline{V}} - (\tilde{\underline{V}} \cdot \tilde{\underline{B}}) \underline{B}_0 \right]$$

$$= -\frac{\underline{B}_0}{4\pi} (\tilde{\underline{V}} \cdot \tilde{\underline{B}})$$

$$\underline{S} = -\frac{\underline{B}_0}{4\pi} \underline{V} \cdot \underline{B}$$

i.e. – energy flows along field

– $\underline{S} \sim \underline{V} \cdot \underline{B}$

$H_c = \int d^3x \, \tilde{\underline{V}} \cdot \tilde{\underline{B}}$ → cross helicity
→ conserved in ideal MHD

Ex.: Show H_c conserved.

→ another way to formulate shear Alfven wave:

since $\tilde{\underline{V}} \perp B_0$ write $\tilde{\underline{V}} = \nabla\phi \times \hat{z}$ ↳ velocity, es potential

$\tilde{\underline{B}} \perp B_0$ $\quad \underline{B} = \nabla A \times \hat{z}$

↳ magnetic potential

i.e. $\underline{E} = \underline{E}_\perp$ so $\tilde{\underline{V}} = \frac{c}{B_0^2} \underline{E} \times \underline{B}_0$ in shear Alfven now, $\frac{\partial \underline{v}}{\partial t} = -\frac{1}{\rho_0} \underline{\nabla} \left(\widetilde{p + \frac{B^2}{8\pi}} \right) + \frac{\underline{B_0} \cdot \underline{\nabla}\, \underline{B}}{4\pi \rho_0}$

as $\underline{\tilde{v}}, \underline{\tilde{B}} \perp B_0$, take $\hat{z} \cdot \underline{\nabla} \times$ $\Rightarrow$

$$\hat{z} \cdot \frac{\partial \underline{\omega}}{\partial t} = 0 + \frac{B_0}{4\pi \rho_0} \frac{\partial}{\partial z} \hat{z} \cdot (\underline{\nabla} \times \underline{B})$$

Now, $\underline{v} = \underline{\nabla} \phi \times \vec{z}$

$= (\partial_y \phi, -\partial_x \phi, 0)$

$\hat{z} \cdot \underline{\nabla} \times \widetilde{\underline{B}} = \frac{4\pi}{c} \tilde{J}_z$

$\underline{\nabla}(\underline{\nabla} \cdot \underline{A}) - \nabla^2 A = +\frac{4\pi}{c} \tilde{J}_z$

$\underline{\omega}_z = \hat{z} \cdot \omega = -\nabla_\perp^2 \phi \rightarrow \vec{z}$ component vorticity

$\Rightarrow$

$$\frac{\partial}{\partial t} \nabla_\perp^2 \phi = \frac{B_0}{4\pi \rho_0} \frac{\partial}{\partial z} \nabla_\perp^2 A$$

magnetic torque

vorticity evolution

$\underline{\nabla} \times (\underline{J} \times \underline{B})$

and $\frac{\partial \tilde{B}}{\partial t} = B_0 \frac{\partial}{\partial z} \underline{v}$ and $\hat{z} \cdot \underline{\nabla} \times$ $\Rightarrow$

$$\frac{\partial}{\partial t} \nabla_\perp^2 A = B_0 \frac{\partial}{\partial z} \nabla_\perp^2 \phi$$

current evolution

$\parallel$ vorticity gradient

observe if "un- $\nabla_\perp^2$", have:

$$\frac{\partial A}{\partial t} - B_0 \frac{\partial}{\partial z} \phi = 0$$

$\Rightarrow$ basically means $E_\parallel = 0$ for Alfven waves.

$$\underline{E} = -\frac{\underline{V} \times \underline{B}_0}{c} \quad \therefore \quad \hat{z} \cdot \frac{\tilde{\underline{V}} \times B_0 \hat{z}}{c} = 0 \checkmark$$

$\therefore$ can write shear Alfven wave equations as

$$\left. \begin{aligned} E_\parallel = 0 &= \frac{\partial A}{\partial t} - B_0 \frac{\partial \phi}{\partial z} = 0 \\ \frac{\partial}{\partial t} \nabla_\perp^2 \phi &= \frac{B_0}{4\pi\rho_0} \frac{\partial}{\partial z} \nabla_\perp^2 A \end{aligned} \right\}$$

$\rightarrow$ example of 'reduced equations'.

Now, need also consider:

$\rightarrow$ $\underline{k} = k\hat{z}$, $\underline{\nabla} \cdot \underline{v} \neq 0$

what happens?

Now, $$\frac{\partial \underline{v}}{\partial t} = -\left(\frac{1}{\rho_0}\right)\nabla\left(\tilde{p} + \frac{B_0\cdot\tilde{B}}{4\pi}\right) + \frac{B_0\cdot\nabla B}{4\pi\rho_0}$$

$$\frac{\partial\tilde{B}}{\partial t} = B_0\cdot\nabla\,\underline{v} - B_0\,\nabla\cdot\tilde{v}$$

$$\underline{k} = k\hat{z} \qquad \nabla\cdot\underline{v} \neq 0$$

$$\Rightarrow \frac{\partial\tilde{v}_z}{\partial t} = -\frac{\partial}{\partial z}\left(\frac{\tilde{p}}{\rho_0}\right) \cancel{- \frac{\partial}{\partial z}\left(\frac{B_0\cdot\tilde{B}}{4\pi\rho_0}\right)} \cancel{+ B_0\frac{\partial}{\partial z}\left(\frac{\tilde{B}_z}{4\pi\rho_0}\right)}$$

and $$\frac{\partial\tilde{B}_z}{\partial t} = \cancel{B_0\,\partial_z\tilde{v}_z} - \cancel{B_0\,\partial_z\tilde{v}_z}$$

∴ all that's left is simple acoustic mode

$$\frac{\partial\tilde{v}_z}{\partial t} = -\frac{\partial}{\partial z}\left(\frac{\tilde{p}}{\rho_0}\right)$$

$$\frac{\tilde{p}}{p_0} = \gamma\,\frac{\tilde{\rho}}{\rho_0} \qquad \text{from} \quad p = p_0\left(\rho/\rho_0\right)^{\gamma}$$

$$\frac{\partial\tilde{\rho}}{\partial t} = -\rho_0\,\nabla\cdot\tilde{v} = -\rho_0\,\partial_z\tilde{v}_z$$

$$\Rightarrow \frac{\partial^2\tilde{\rho}}{\partial t^2} = \gamma\,\frac{p_0}{\rho_0}\,\frac{\partial^2}{\partial z^2}\tilde{\rho}$$

$$\Rightarrow \quad \omega^2 = c_s^2 k_z^2 \quad , \quad c_s^2 = \underset{\text{"stiffness"}}{\gamma}\ \overset{\text{energy density}}{P}/\rho_0$$

$\rightarrow$ $\underline{k} = k\,\vec{x}$ — Perpendicular Propagation

Now $\underline{B} = B_0\,\hat{z}$, so

$\rightarrow$ $\underline{k} = k\,\vec{x}$ must compress magnetic field

$\therefore$

$\rightarrow$ no incompressible cross-field propagation is possible

Now

$$\frac{\partial \underline{V}}{\partial t} = -\frac{\underline{\nabla}}{\rho_0}\widetilde{\left(P + \frac{B^2}{8\pi}\right)} + \cancel{\frac{B_0 \cdot \underline{\nabla}\ \underline{\tilde{B}}}{4\pi\rho_0}}$$

and

$$\frac{\partial}{\partial t} B/\rho = \cancel{\frac{B_0 \cdot \underline{\nabla}\ \underline{\tilde{V}}}{\rho_0}} \quad \text{— freezing in}$$

so can take short-cut via:

$$\frac{d}{dt} B/\rho = 0 \quad \Rightarrow \quad \tilde{B} = B_0 \frac{\tilde{\rho}}{\rho_0}$$

$$\frac{\partial V}{\partial t} = -\frac{1}{\rho_0} \nabla\left(\overset{\text{thermal}}{\overbrace{P_T}} + \underset{\text{magnetic}}{\underbrace{P_B}}\right)$$

$$P_T = P_0 (\rho/\rho_0)^\gamma , \quad \tilde{P}_T = \gamma P_0 (\tilde{\rho}/\rho_0)$$

$$P_B = B^2/8\pi , \quad \tilde{P}_B = 2\frac{B_0^2}{8\pi}(\tilde{\rho}/\rho_0)$$

(i.e. "γ_{eff}" $= 2$, for field)

$$\frac{\partial}{\partial t}(\nabla\cdot\tilde{V}) = -\nabla^2\left[\frac{\gamma P_0}{\rho_0} + \frac{2B_0^2}{8\pi\rho_0}\right]\frac{\tilde{\rho}}{\rho_0}$$

but $\nabla\cdot V = -\frac{\partial}{\partial t}\frac{\tilde{\rho}}{\rho_0}$

$$\Rightarrow \frac{\partial^2}{\partial t^2}(\tilde{\rho}/\rho_0) = \nabla^2\left[\frac{\gamma P_0}{\rho_0} + \frac{2B_0}{8\pi\rho_0}\right](\tilde{\rho}/\rho_0)$$

$$= \nabla^2\left[C_s^2 + V_A^2\right](\tilde{\rho}/\rho_0)$$

$$\boxed{\omega^2 = k_\perp^2(C_s^2 + V_A^2)}$$

$\Rightarrow$ "magnetosonic"
or
"compressional Alfven wave"

N.B. :

→ magnetosonic wave has $\omega^2 = c_s^2 + V_A^2$
⇔ combines acoustic, magnetic speeds
→ always faster (higher phase speed) than shear Alfven or acoustic mode.

i.e. $\underline{k} = \underline{k}_\perp$ magnetosonic wave is "fastest" MHD wave

→ recalling class discussion ? ⇒ how reconcile ?

- magnetosonic wave carried by field energy density → $B_0^2/8\pi$

Yet

- $V_{magn}^2 = V_A^2$, as in shear Alfven, which is carried by magnetic tension $B_0^2/4\pi\rho_0$.

Resolution : Freezing-in condition ⇒ $\underline{B}/\rho = \text{const}$, here

⇒ $\gamma_{eff} = 2$

i.e. freezing-in condition ⇒ field is stiff — indeed stiffer than gas, $\gamma = 5/3$ — acoustic medium i.e. $C_S^2 = c_s^2 + c_B^2$

$$= \frac{dP_{Th}}{d\rho} + \frac{dP_B}{d\rho}$$

$$= \gamma \frac{P_{Th_0}}{\rho_0} + 2\frac{P_B}{\rho_0}$$

i.e. for $\beta = P_{Th}/P_B = 1 \Rightarrow c_B^2 > c_s^2$

So can summarize simple cases:

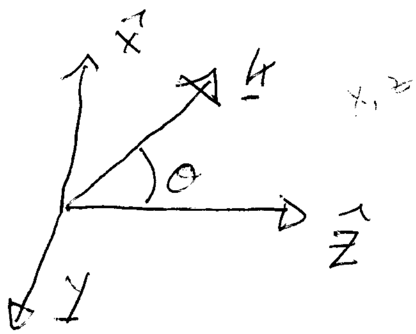
	$\nabla \cdot \underline{V} = 0$	$\nabla \cdot \underline{V} \neq 0$
$\underline{k} = \underline{k}_{\parallel}$	$\omega^2 = k_{\parallel}^2 V_A^2$ shear Alfven	$\omega^2 = k_{\parallel}^2 c_s^2$ acoustic
$\underline{k} = \underline{k}_{\perp}$	X	$\omega^2 = k_{\perp}^2 (c_s^2 + V_A^2)$ magnetosonic wave

Note that magnetosonic is 'fastest' of waves.

(ii.) Full Crank — Read Kulsrud, Chapt. 5

Now, consider full crank, for arbitrary $\underline{k}$.

geometry:



$$\left\{ \begin{array}{l} \rho_0 = \rho_0 = \text{const} \\ \underline{B} = B_0 \hat{z} \end{array} \right.$$

have MHD equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{V}) = 0$$

$$\rho \frac{d\underline{V}}{dt} = -\underline{\nabla} p + \frac{\underline{J} \times \underline{B}}{c}$$

$$\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times (\underline{V} \times \underline{B})$$

$$\frac{d}{dt}\left(p/\rho^\gamma\right) = 0 \quad \Rightarrow \quad \frac{1}{p}\frac{dp}{dt} - \frac{\gamma}{\rho}\frac{d\rho}{dt} = 0$$

and continuity $\Rightarrow$

$$\frac{1}{p}\frac{dp}{dt} = -\gamma \, \underline{\nabla} \cdot \underline{V}$$

Now, convenient to write $\boxed{\underline{V}(\underline{x},t) = \frac{\partial \underline{\xi}}{\partial t}(\underline{x},t)}$

$\underline{\xi}(\underline{x},t) \equiv$ displacement of fluid element, originally at $\underline{x}$ at t

$\Rightarrow$ with linearization $\tilde{\underline{V}} = \frac{\partial \underline{\xi}}{\partial t}$, $\rho = \rho_0 + \delta\rho$, etc. :

$$\delta\rho = -\rho_0 \underline{\nabla}\cdot\underline{\xi}$$

$$\delta P = -\gamma p_0 \underline{\nabla}\cdot\underline{\xi}$$

$$\delta \underline{B} = \nabla \times (\underline{\xi} \times \underline{B}_0)$$

$$\rho_0 \frac{\partial^2 \underline{\xi}}{\partial t^2} = -\underline{\nabla}\delta p + \left(\frac{\delta \underline{J} \times \underline{B}_0}{c}\right)$$

so can assemble the pieces, assuming $\underline{\xi} = \underline{\xi}_{\underline{k}} e^{i(k \cdot x - \omega t)}$ and omitting subscript $\Rightarrow$

$\delta J = \underline{\nabla} \times \delta \underline{B}$

$$\boxed{-\rho_0 \omega^2 \underline{\xi} = -\gamma p_0 \underline{k}(\underline{k}\cdot\underline{\xi}) - \frac{1}{4\pi}\left[\underline{k} \times (\underline{k} \times (\underline{\xi} \times \underline{B}_0))\right] \times \underline{B}_0}$$

from induction

- eigenmode equation for arbitrary displacement
- note as $\underline{\xi}$ is a 3 component vector above is $\underline{\underline{3}}$ linearly coupled equations. ω^2 is the eigenvalue! So ...

so - solution is det $|3\times3|$ $\Rightarrow$ cubic equation for ω^2. $\Rightarrow$ expect $\underline{\underline{3}}$ waves.

N.B.: Based on simple cases, what might these be?

$$\boxed{-\rho_0\omega^2\underline{\xi} = -\gamma P_0\underline{k}(\underline{k}\cdot\underline{\xi}) - \frac{1}{4\pi}\left\{\underline{k}\times[\underline{k}\times\underline{\xi}\times\underline{B_0}]\right\}\times\underline{B_0}}$$

$\Rightarrow$ the 3 waves are, for the obvious profound reason, called the "fast", "slow" and "intermediate" waves...

- now, <u>choose</u>: $\begin{cases}\underline{k} = k(\sin\theta\,\hat{x} + \cos\theta\,\hat{z}) \\ \underline{\xi} = \xi\,\hat{y}\end{cases}$ oblique in xz plane

i.e. $\underline{k}\cdot\underline{\xi} = 0 \Rightarrow \underline{\nabla}\cdot\underline{V} = 0$

$\Rightarrow$ "intermediate wave" $\rightarrow$ clearly <u>shear</u> <u>Alfven</u>

now $\underline{k}\cdot\underline{\xi} = 0$

and crank $\Rightarrow$ $\left[\underline{k}\times[\underline{k}\times(\underline{\xi}\times\underline{B_0})]\right]\times\frac{\underline{B_0}}{4\pi}$

$$= \frac{(\underline{k}\cdot\underline{B_0})}{4\pi}[\underline{k}\times(\underline{\xi}\times\underline{B_0})]$$

$$= \frac{(\underline{k}\cdot\underline{B_0})^2}{4\pi}\underline{\xi}$$

so $\quad -\rho_0 \omega^2 \underline{\xi} = -\frac{(\underline{k}\cdot \underline{B}_0)^2}{4\pi} \underline{\xi}$

$\underline{\xi} = \xi_y \hat{y}$

$\Rightarrow \quad \omega^2 = k_{\parallel}^2 V_A^2 \quad$ with $\underline{\xi} = \xi_y \hat{y}$

shear Alfven $\rightarrow$ physical properties as before

$\therefore$ "intermediate wave" is shear Alfven

$\underline{\underline{\text{so}}}$ "fast wave" must connect to magnetosonic

$\therefore$ "slow wave" must connect to acoustic

$(c_s^2 < V_A^2)$

Let's see ...

— fast and slow waves:

again: $\quad \underline{k} = k(\sin\theta\, \hat{x} + \cos\theta\, \hat{z})$

$$\underline{\xi} = \xi_x \hat{x} + \xi_z \hat{z}$$

point here is that $\underline{k}\cdot\underline{\xi} \neq 0 \Rightarrow$ unlike intermediate, these are compressional so now, crank ⇒

$$\frac{1}{4\pi}\left\{ \underline{k}\times\left[\underline{k}\times(\underline{\xi}\times\underline{B}_0)\right]\right\}\times\underline{B}_0 = -\frac{k^2B_0^2}{4\pi}\,\xi_x\,\hat{x}$$

and

$$-\nabla p_1 = -\gamma P_0\,\underline{k}\,(\underline{k}\cdot\underline{\xi})$$

so

$$-\frac{\partial p_1}{\partial x} = -k^2\gamma P_0\left(\sin^2\theta\,\xi_x + \sin\theta\cos\theta\,\xi_z\right)$$

$$-\frac{\partial p_1}{\partial z} = -k^2\gamma P_0\left(\sin\theta\cos\theta\,\xi_x + \cos^2\theta\,\xi_z\right)$$

now, defining $\left.\begin{array}{l} c_s^2 = \gamma P_0/\rho_0 \\ v_A^2 = B_0^2/4\pi\rho_0 \end{array}\right\}$ as usual ⇒

$$\boxed{\begin{array}{l} -\omega^2\xi_x = -k^2\left(c_s^2\sin^2\theta + v_A^2\right)\xi_x - k^2c_s^2\sin\theta\cos\theta\,\xi_z \\ -\omega^2\xi_z = -k^2c_s^2\sin\theta\cos\theta\,\xi_x - k^2c_s^2\cos^2\theta\,\xi_z \end{array}}$$

⇒ coupled equations for ξ_x, ξ_z

⇒ standard crank gives:

$$\begin{vmatrix} k^2v_A^2 + k^2c_s^2\sin^2\theta - \omega^2, & k^2c_s^2\sin\theta\cos\theta \\ k^2c_s^2\sin\theta\cos\theta, & k^2c_s^2\cos^2\theta - \omega^2 \end{vmatrix} = 0$$

and $$\boxed{\omega^4 - k^2\left(C_s^2 + V_A^2\right)\omega^2 + k^4 C_s^2 V_A^2 \cos\theta = 0}$$

is "the dispersion relation".

Now can solve as:

$$\boxed{\frac{\omega^2}{k^2} = \frac{V_A^2 + C_s^2}{2} \pm \frac{1}{2}\left[\left(V_A^2 - C_s^2\right)^2 + 4 C_s^2 V_A^2 \sin^2\theta\right]^{1/2}}$$

→ upper root → "fast" wave
→ lower root → "slow" wave.

now check:

$\sin\theta = 0 \Rightarrow \underline{k} = k\,\hat{z}$

$$\frac{\omega^2}{k^2} = \frac{V_A^2 + C_s^2}{2} \pm \frac{\left(V_A^2 - C_s^2\right)}{2} \longrightarrow \begin{array}{l} V_A^2 \rightarrow \text{Alfven} \\ C_s^2 \rightarrow \text{acoustic} \end{array}$$

$\sin\theta = 1 \Rightarrow \underline{k} = k\hat{x}$

$$\frac{\omega^2}{k^2} = \frac{V_A^2 + C_s^2}{2} \pm \frac{1}{2}\left[\left(V_A^2\right)^2 + \left(C_s^2\right)^2 - 2V_A^2 C_s^2 + 4 C_s^2 V_A^2\right]^{1/2}$$

$$= \frac{V_A^2 + C_s^2}{2} \pm \frac{1}{2}\left[\left(V_A^2 + C_s^2\right)^2\right]^{1/2} = \begin{cases} 0 \\ V_A^2 + C_s^2 \\ \quad\rightarrow \text{Magnetosonic wave.} \end{cases}$$

Note: can observe:

- for $\perp$ propagation, fast wave $\Leftrightarrow$ magnetosonic wave
[slow = intermediate wave : $\omega^2 = 0$]
- for $\parallel$ propagation, fast $\Leftrightarrow$ Alfven ($\beta < 1$)
[fast $\Rightarrow$ intermediate] slow $\Leftrightarrow$ acoustic
($\beta > 1$ vice versa)
- always have $v_{ph_{slow}}^2 \leq v_{ph_{int}}^2 \leq v_{ph_{fast}}^2$

$\Rightarrow$ have general result that polarizations of fast and slow modes are orthogonal

can show via:

$\mapsto$ matrix from eqns $\Leftrightarrow$ 2x2

$$-\rho \omega_s^2 \underline{\xi_s} = \underline{\underline{M}} \cdot \underline{\xi_s} \qquad (1)$$

$$-\rho \omega_f^2 \underline{\xi_f} = \underline{\underline{M}} \cdot \underline{\xi_f} \qquad (2)$$

$\underline{\xi_f} \cdot (1) - \underline{\xi_s} \cdot (2) \Rightarrow$

$$-\rho (\omega_s^2 - \omega_f^2) \underline{\xi_s} \cdot \underline{\xi_f} = \xi_f \cdot \underline{\underline{M}} \cdot \xi_s - \xi_s \cdot \underline{\underline{M}} \cdot \xi_f$$

but: recall from determinant

$$\underline{\underline{M}} = -\begin{bmatrix} k^2 V_A^2 + k^2 c_s^2 \sin^2\theta , & k^2 c_s^2 \sin\theta\cos\theta \\ k^2 c_s^2 \sin\theta\cos\theta , & k^2 c_s^2 \cos^2\theta \end{bmatrix}$$

and $\underline{\underline{M}}^T = \underline{\underline{M}}$ so $\underline{\underline{M}}$ self-adjoint!

$\Rightarrow$

$$\underline{\xi}_f \cdot \underline{\underline{M}} \cdot \underline{\xi}_s = \underline{\xi}_s \cdot \underline{\underline{M}} \cdot \underline{\xi}_f$$

$\hookrightarrow$ {important structural property in linear MHD

so $\underline{\xi}_f \cdot \underline{\xi}_s = 0$.

$\rightarrow$ to yet furthur elucidate the waves, can consider two limits $\beta \ll 1 \rightarrow c_s^2/V_A^2 \ll 1$
$\beta \gg 1 \rightarrow c_s^2/V_A^2 \gg 1$.

a) for $c_s^2 \gg V_A^2$,

(pressure $\leftrightarrow$ springiness) (to p. 85)

l. order $\omega_f^2 = k^2 c_s^2$, $\omega_s = 0$

1st ord. $\dfrac{\omega_f}{k} \sim c_s + \dfrac{V_A^2 \sin^2\theta}{2c_s}$,

$\tilde{\underline{\xi}} \parallel \underline{k}$

(note $\xi_f \cdot \xi_s = 0$)

$$\frac{\omega_s^2}{k^2} \approx V_A^2 \cos^2\theta$$

$\tilde{\underline{\xi}} \perp \underline{k}$

(otherwise $\tilde{\rho} \rightarrow$ higher ω)

b) for $C_s^2 \ll V_A^2$, (springiness → B field)

$$\frac{\omega_f^2}{k^2} \cong V_A^2 + C_s^2 \sin^2\theta$$

$\underline{\varepsilon} \perp \underline{B_o}$
(or no "springiness" to drive fast motion in ~~parallel~~ $\perp$ dir.)

$$\frac{\omega_s^2}{k^2} \simeq C_s^2 \cos^2\theta$$

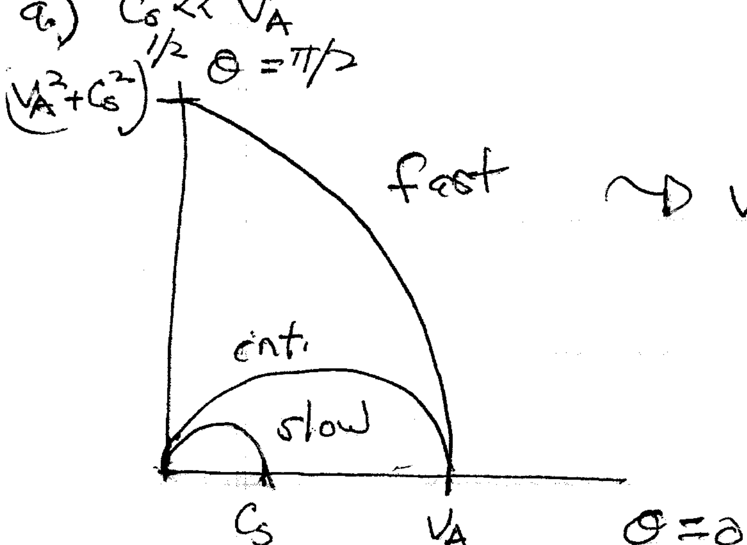
$\varepsilon \parallel B_o$
(otherwise, if $\varepsilon \perp B_o$ → get Alfven)

and again, $\underline{\varepsilon}_s \cdot \underline{\varepsilon}_f = 0$

→ now, can sum up this slow, intermediate, fast story in the Fredriks Diagram

consider $C_s \ll V_A$, $C_s \gg V_A$

a) $C_s \ll V_A$



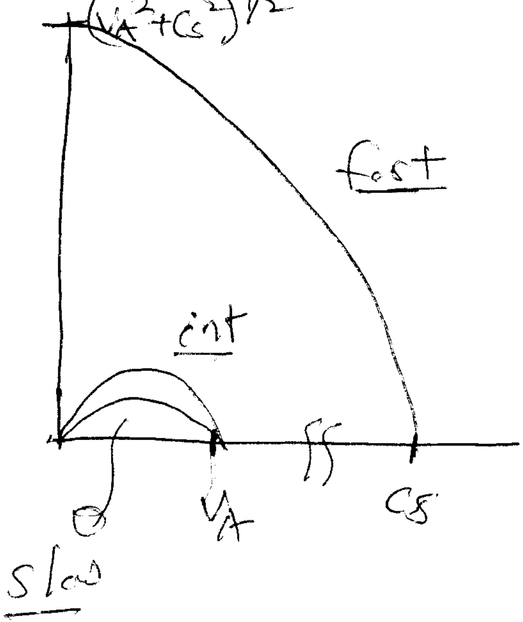
→ V_{phase} vs θ for:

fast → magnetosonic at $\perp$
Alfven at $\parallel$

int → Alfven at $\parallel$
nothing at $\perp$

slow → acoustic (parallel) at $\parallel$
nothing at $\perp$.

b.) $V_A^2 \ll C_s^2$



again:

fast → magnetosonic at ⊥
Alfven at ∥

int. → Alfven at ∥
nothing at perp.

slow → Alfven at ∥
nothing at ⊥

→ now, observe the following:

→ 3 components $\underline{v}$
→ 2 component $\underline{B}$ $(\underline{\nabla}\cdot\underline{B}=0)$
→ ρ, P

⇒

∴ 7 fields

but 6 waves → 2 each $\begin{cases} \text{fast} \\ \text{intermediate} \\ \text{slow} \end{cases}$

$\omega^2 = \_\_\_$

so, 1 missing mode! → entropy mode!

i.e. $S = T \ln(P/\rho^{\gamma})$

and assumed $P_1/P_0 = \gamma \rho_1/\rho_0$

if relax ⇒ entropy wave
relevant in shocks

$$\begin{cases} \delta\rho \neq 0, & \text{all else} = 0 \\ \omega = 0. \end{cases}$$

→ some concluding philosophy ⇒ what is the moral of this story of the trip to the zoo of MHD waves?

- even for ~ simple dynamical model like ideal MHD, even minimal anisotropy introduces great complexity!

- signal propagation
  - parameter dependent
  - anisotropic
  - has definite polarization

- important to understand
  - magnetic pressure
  - magnetic tension
  - thermal pressure

as origins of anisotropic restoring force in waves.